ON THE COBAR CONSTRUCTION OF A BIALGEBRA

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ABSTRACT. We show that the cobar construction of a DG-bialgebra is a homotopy G-algebra. This implies that the bar construction of this cobar is a DG-bialgebra as well.

1. INTRODUCTION

The cobar construction ΩC of a DG-coalgebra $(C, d : C \to C, \Delta : C \to C \otimes C)$ is, by definition, a DG-algebra. Suppose now that C is additionally equipped with a multiplication $\mu : C \otimes C \to C$ turning (C, d, Δ, μ) into a DG-bialgebra. How this multiplication reflects on the cobar construction ΩC?

It was shown by Adams [1] that in mod 2 situation in this case the multiplication of ΩC is homotopy commutative: there exists a $\smile_1$ product

$$\smile_1 : \Omega C \otimes \Omega C \to \Omega C$$

which satisfies the standard condition

$$d(a \smile_1 b) = da \smile_1 b + a \smile_1 db + a \cdot b + b \cdot a, \qquad (1)$$

(since we work mod 2 the signes are ignored in whole paper).

In this note we show that this $\smile_1$ gives rise to a sequence of operations

$$E_{1,k} : \Omega C \otimes (\Omega C)^{\otimes k} \to \Omega C,\ \ k = 1, 2, 3, ...$$

which form on the cobar construction ΩC of a DG-bialgebra a structure of *homotopy G-algebra* (hGa) in the sense of Gerstenhaber and Voronov [8].

There are two remarkable examples of homotopy G-algebras.

The first one is the cochain complex of 1-reduced simplicial set $C^*(X)$. The operations $E_{1,k}$ here are dual to cooperations defined by Baues in [2], and the starting operation $E_{1,1}$ is the classical Steenrod's $\smile_1$ product.

The second example is the Hochschild cochain complex $C^*(U, U)$ of an associative algebra U. The operations $E_{1,k}$ here were defined in [11] with the purpose to describe $A(\infty)$-algebras in terms of Hochschild cochains although the properties of those operations which where used as defining ones for the notion of homotopy G-algebra in [8] did not appear there. These operations where defined also in [9]. Again the starting operation $E_{1,1}$ is the classical Gerstenhaber's circle product which is sort of $\smile_1$-product in the Hochschild complex.

In this paper we present the third example of homotopy G-algebra: we construct the operations $E_{1,k}$ on the cobar construction ΩC of a DG-bialgebra C, and the starting operation $E_{1,1}$ is again classical, it is Adams's $\smile_1$-product.

The notion of hGa was introduced in [8] as an additional structure on a DG-algebra $(A, d, \cdot)$ that induces a Gerstenhaber algebra structure on homology. The

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source of the defining identities and the main example was Hochschild cochain complex $C^*(U,U)$. Another point of view is that hGa is a particular case of $B(\infty)$-algebra. This is an additional structure on a DG-algebra $(A,d,\cdot)$ that induces a DG-bialgebra structure on the bar construction BA.

We emphasize the third aspect of hGa: this is a structure which measures the noncommutativity of A. There exists the classical tool which measures the noncommutativity of a DG-algebra $(A,d,\cdot)$, namely the Steenrod's $\smile_1$ product, satisfying the condition (1). The existence of such $\smile_1$ guarantees the commutativity of $H(A)$, but $\smile_1$ product satisfying just the condition (1) is too poor for most of applications. In many constructions some deeper properties of $\smile_1$ are needed, for example the compatibility with the dot product of A (the Hirsch formula)

$$(a\cdot b)\smile_1 c + a\cdot(b\smile_1 c) + (a\smile_1 c)\cdot b = 0. \tag{2}$$

For a hGa $(A,d,\cdot,\{E_{1,k}\})$ the starting operation $E_{1,1}$ is a kind of $\smile_1$ product: it satisfies the conditions (1) and (2). As for the symmetric expression

$$a\smile_1(b\cdot c) + b\cdot(a\smile_1 c) + (a\smile_1 b)\cdot c,$$

it is just *homotopical to zero* and the appropriate homotopy is the operation $E_{1,2}$. The defining conditions of a hGa which satisfy higher operations $E_{1,k}$ can be regarded as generalized Hirsch formulas. So we can say that a hGa is a DG-algebra with a "good" $\smile_1$ product.

2. Notation and preliminaries

We work over Z_2. For a graded Z_2-module M we denote by sM the suspension of M, i.e. $(sM)^i = M^{i-1}$. Respectively $s^{-1}M$ denotes the desuspension of M, i.e. $(s^{-1}M)^i = M^{i+1}$.

A *differential graded algebra* (DG-algebra) is a graded R-module $C=\{C^i\},\ i\in Z$, with an associative multiplication $\mu: C^i\otimes C^j\to C^{i+j}$ and a homomorphism (a differential) $d: C^i\to C^{i+1}$ with $d^2=0$ and satisfying the Leibniz rule $d(x\cdot y) = dx\cdot y + x\cdot dy$, where $x\cdot y = \mu(x\otimes y)$. We assume that a DG-algebra contains a unit $1\in C^0$. A non-negatively graded DG-algebra C is *connected* if $C^0 = Z_2$. A connected DG-algebra C is *n-reduced* if $C^i = 0, 1\le i\le n$. A DG-algebra is *commutative* if $\mu = \mu T$, where $T(x\otimes y) = y\otimes x$.

A *differential graded coalgebra* (DG-coalgebra) is a graded Z_2-module $C = \{C_i\},\ i\in Z$, with a coassociative comultiplication $\Delta: C\to C\otimes C$ and a homomorphism (a differential) $d: C_i\to C_{i+1}$ with $d^2=0$ and satisfying $\Delta d = (d\otimes id + id\otimes d)\Delta$. A DG-coalgebra C is assumed to have a counit $\epsilon: C\to Z_2$, $(\epsilon\otimes id)\Delta = (id\otimes\epsilon)\Delta = id$. A non-negatively graded dgc C is *connected* if $C_0 = Z_2$. A connected DG-coalgebra C is *n-reduced* if $C_i = 0, 1\le i\le n$.

A *differential graded bialgebra* (DG-bialgebra) (C,d,μ,Δ) is a DG-coalgebra (C,d,Δ) with a morphism of DG-coalgebras $\mu: C\otimes C\to C$ turning (C,d,μ) into a DG-algebra.

2.1. **Cobar and Bar constructions.** Let M be a graded Z_2-vector space with $M^{i\le 0}=0$ and let $T(M)$ be the tensor algebra of M, i.e. $T(M) = \oplus_{i=0}^{\infty} M^{\otimes i}$. $T(M)$ is a free graded algebra: for a graded algebra A and a homomorphism $\alpha: M\to A$ of degree zero there exists its *multiplicative extension*, a unique morphism of graded algebras $f_\alpha: T(M)\to A$ such that $f_\alpha(a) = \alpha(a)$. The map f_α is given by $f_\alpha(a_1\otimes ...\otimes a_n) = \alpha(a_1)\cdot ...\cdot\alpha(a_n)$.

Dually, let $T^c(M)$ be the tensor coalgebra of M, i.e. $T^c(M)=\oplus_{i=0}^{\infty}M^{\otimes i}$, and the comultiplication $\nabla : T^c(M)\to T^c(M)\otimes T^c(M)$ is given by

$$\nabla(a_1\otimes ...\otimes a_n)=\sum_{k=0}^{n}(a_1\otimes ...\otimes a_k)\otimes(a_{k+1}\otimes ...\otimes a_n).$$

$(T^c(M),\nabla)$ is a cofree graded coalgebra: for a graded coalgebra C and a homomorphism $\beta : C\to M$ of degree zero there exists its *comultiplicative extension*, a unique morphism of graded coalgebras $g_\beta : C\to T^c(M)$ such that $p_1g_\beta=\beta$, here $p_1 : T^c(M)\to M$ is the clear projection. The map g_β is given by

$$g_\beta(c)=\sum_n \beta(c^{(1)})\otimes ...\otimes \beta(c^{(n)}),$$

where $\Delta^n(c)=c^{(1)}\otimes ...\otimes c^{(n)}$ and $\Delta^n : C\to C^{\otimes n}$ is n-th iteration of the diagonal $\Delta : C\to C\otimes C$, i.e. $\Delta^1=id,\ \Delta^2=\Delta,\ \Delta^n=(\Delta^{n-1}\otimes id)\Delta$.

Let (C,d_C,Δ) be a connected DG-coalgebra and $\Delta=id\otimes 1+1\otimes id+\Delta'$. The (reduced) *cobar construction* ΩC on C is a DG-algebra whose underlying graded algebra is $T(sC^{>0})$. An element $(sc_1\otimes ...\otimes sc_n)\in(sC)^{\otimes n}\subset T(sC^{>0})$ is denoted by $[c_1,...,c_n]\in\Omega C$. The differential on ΩC is the sum $d=d_1+d_2$ which for a generator $[c]\in\Omega C$ is defined by $d_1[c]=[d_C(c)]$ and $d_2[c]=\sum[c',c'']$ for $\Delta'(c)=\sum c'\otimes c''$, and extended as a derivation.

Let (A,d_A,μ) be a 1-reduced DG-algebra. The (reduced) *bar construction* BA on A is a DG-coalgebra whose underlying graded coalgebra is $T^c(s^{-1}A^{>0})$. Again an element $(s^{-1}a_1\otimes ...\otimes s^{-1}a_n)\in(s^{-1}A)^{\otimes n}\subset T^c(s^{-1}A^{>0})$ we denote as $[a_1,...,a_n]\in BA$. The differential of BA is the sum $d=d_1+d_2$ which for an element $[a_1,...a_n]\in BA$ is defined by

$$d_1[a_1,...,a_n]=\sum_{i=1}^{n}[a_1,...,d_Aa_i,...,a_n],d_2[a_1,...,a_n]=\sum_{i=1}^{n-1}[a_1,...,a_i\cdot a_{i+1},...,a_n].$$

2.2. **Twisting cochains.** Let (C,d,Δ) be a dgc, (A,d,μ) a dga. A twisting cochain [5] is a homomorphism $\tau : C\to A$ of degree $+1$ satisfying the Browns' condition

$$d\tau+\tau d=\tau\smile\tau, \tag{3}$$

where $\tau\smile\tau'=\mu_A(\tau\otimes\tau')\Delta$. We denote by $T(C,A)$ the set of all twisting cochains $\tau : C\to A$.

There are universal twisting cochains $C\to\Omega C$ and $BA\to A$ being clear inclusion and projection respectively. Here are essential consequences of the condition (3):
(i) The multiplicative extension $f_\tau : \Omega C\to A$ is a map of DG-algebras, so there is a bijection $T(C,A)\leftrightarrow Hom_{DG-Alg}(\Omega C,A)$;
(ii) The comultiplicative extension $g_\tau : C\to BA$ is a map of DG0coalgebras, so there is a bijection $T(C,A)\leftrightarrow Hom_{DG-Coalg}(C,BA)$.

3. Homotopy G-algebras

3.1. **Products in the bar construction.** Let $(A,d,\cdot)$ be a 1-reduced DG-algebra and BA it's bar construction. We are interested in the structure of a multiplication

$$\mu : BA\otimes BA\to BA,$$

turning BA into a DG-bialgebra, i.e. we require that

(i) μ is a DG-coalgebra map;
(ii) is associative;
(iii) has the unit element $1_\Lambda \in \Lambda \subset BA$.

Because of the cofreeness of the tensor coalgebra $BA = T^c(s^{-1}A)$, a map of graded coalgebras

$$\mu : BA \otimes BA \to BA$$

is uniquely determined by the projection of degree $+1$

$$E = pr \cdot \mu : BA \otimes BA \to BA \to A.$$

Conversly, a homomorphism $E : BA \otimes BA \to A$ of degree $+1$ determines it's coextension, a graded coalgebra map $\mu_E : BA \otimes BA \to BA$ given by

$$\mu_E = \sum_{k=0}^{\infty} (E \otimes ... \otimes E)\nabla^k_{BA\otimes BA},$$

where $\nabla^k_{BA\otimes BA} : BA \otimes BA \to (BA \otimes BA)^{\otimes k}$ is the k-fold iteration of the standard coproduct of tensor product of coalgebras

$$\nabla_{BA\otimes BA} = (id \otimes T \otimes id)(\nabla \otimes \nabla) : BA \otimes BA \to (BA \otimes BA)^{\otimes 2}.$$

The map μ_E is a *chain map* (i.e. it is a map of DG-coalgebras) if and only if E is a twisting cochain in the sense of E. Brown, i.e. satisfies the condition

$$dE + Ed_{BA\otimes BA} = E \smile E. \tag{4}$$

Indeed, again because of the cofreeness of the tensor coalgebra $BA = T^c(s^{-1}A)$ the condition $d_{BA}\mu_E = \mu_E d_{BA\otimes BA}$ is satisfied if and only if it is satisfied after the projection on A, i.e. if $pr \cdot d_{BA}\mu_E = pr \cdot \mu_E d_{BA\otimes BA}$ but this condition is nothing else than the Brown's condition (4).

The same argument shows that the product μ_E *is associative* if and only if $pr \cdot \mu_E(\mu_E \otimes id) = pr \cdot \mu_E(id \otimes \mu_E)$, or, having in mind $E = pr \cdot \mu_E$

$$E(\mu_E \otimes id) = E(id \otimes \mu_E). \tag{5}$$

A homomorphism $E : BA \otimes BA \to A$ consists of *components*

$$\{\bar{E}_{p,q} : (s^{-1}A)^{\otimes p} \otimes (s^{-1}A)^{\otimes q} \to A,\ p, q = 0, 1, 2, ...\},$$

where $\bar{E}_{pq}$ is the restriction of E on $(s^{-1}A)^{\otimes p} \otimes (s^{-1}A)^{\otimes q}$. Each component $\bar{E}_{p,q}$ can be regarded as an operation

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A,\ p, q = 0, 1, 2, ...$$

The value of $E_{p,q}$ on the element $(a_1 \otimes ... \otimes a_p) \otimes (b_1 \otimes ... \otimes b_q)$ we denote by $E_{p,q}(a_1, ..., a_p; b_1..., b_q)$.

It is not hard to check that the multiplication μ_E induced by E (or equivalently by a collection of multioperations $\{E_{p,q}\}$) *has the unit* $1_\Lambda \in \Lambda \subset BA$ if and only if

$$E_{0,1} = E_{1,0} = id;\quad E_{0,k} = E_{k,0} = 0,\ k > 1. \tag{6}$$

So we can summarize:

Proposition 1. *The multiplication μ_E induced by a collection of multioperations $\{E_{p,q}\}$ turns BA into a DG-bialgebra, i.e. satisfies (i-iii), if and only if the conditions (4), (5), and (6) are satisfied.*

Let us interpret the condition (4) in terms of the components E_{pq}.

The restriction of (4) on $A\otimes A$ gives

$$dE_{1,1}(a;b)+E_{1,1}(da;b)+E_{1,1}(a;db)=a\cdot b+b\cdot a. \tag{7}$$

This condition coincides with the condition (1), i.e. the operation $E_{1,1}$ is sort of $\smile_1$ product, which measures the noncommutativity of A. Below we denote $E_{1,1}(a;b)=a\smile_1 b$.

The restriction on $A^{\otimes 2}\otimes A$ gives

$$\begin{gathered}dE_{2,1}(a,b;c)+E_{2,1}(da,b;c)+E_{2,1}(a,db;c)+E_{2,1}(a,b;dc)=\\(a\cdot b)\smile_1 c+a\cdot(b\smile_1 c)+(a\smile_1 c)\cdot b,\end{gathered} \tag{8}$$

this means, that this $\smile_1$ satisfies the *left Hirsch formula* (2) up to homotopy and the appropriate homotopy is the operation $E_{2,1}$.

The restriction on $A\otimes A^{\otimes 2}$ gives:

$$\begin{gathered}dE_{1,2}(a;b,c)+E_{1,2}(da;b,c)+E_{1,2}(a;db,c)+E_{1,2}(a;b,dc)=\\a\smile_1(b\cdot c)+(a\smile_1 b)\cdot c+b\cdot(a\smile_1 c),\end{gathered} \tag{9}$$

this means, that this $\smile_1$ satisfies the *right Hirsch formula* (2) up to homotopy and the appropriate homotopy is the operation $E_{1,2}$.

Generally the restriction of (4) on $A^{\otimes m}\otimes A^{\otimes n}$ gives:

$$\begin{gathered}dE_{m,n}(a_1,...,a_m;b_1,...,b_n)+\sum_i E_{m,n}(a_1,...,da_i,...,a_m;b_1,..,b_n)\\+\sum_i E_{m,n}(a_1,...,a_m;b_1,...,db_i,..,b_n)=\\a_1\cdot E_{m-1,n}(a_2,...,a_m;b_1,...,b_n)+E_{m-1,n}(a_1,...,a_{m-1};b_1,...,b_n)\cdot a_m\\+b_1\cdot E_{m,n-1}(a_1,...,a_m;b_2,...,b_n)+E_{m,n-1}(a_1,...,a_m;b_1,...,b_{n-1})\cdot b_m+\\\sum_i E_{m-1,n}(a_1,...,a_i\cdot a_{i+1},...,a_m;b_1,...,b_n)+\\\sum_i E_{m,n-1}(a_1,...,a_m;b_1,...,b_i\cdot b_{i+1},...,b_n)+\\\sum_{p=1}^{m-1}\sum_{q=1}^{n-1}E_{p,q}(a_1,...,a_p;b_1,..,b_q)\cdot E_{m-p,n-q}(a_{p+1},...,a_m;b_{q+1},...,b_n).\end{gathered} \tag{10}$$

Now let us interpret the associativity condition (5) in terms of the components $E_{p,q}$. The restriction of (5) on $A\otimes A\otimes A$ gives

$$\begin{gathered}(a\smile_1 b)\smile_1 c+a\smile_1(b\smile_1 c)=E_{1,2}(a;b,c)+E_{1,2}(a;c,b)+\\E_{2,1}(a,b;c)+E_{2,1}(b,a;c).\end{gathered} \tag{11}$$

Generally the restriction of (5) on $A^{\otimes k}\otimes A^{\otimes l}\otimes A^{\otimes m}$ gives

$$\begin{gathered}\sum_{r=1}^{l+m}\sum_{l_1+...+l_r=l,m_1+...+m_r=m}\\E_{k,r}(a_1,...,a_k;E_{l_1,m_1}(b_1,...,b_{l_1};c_1,...,c_{m_1}),...,\\E_{l_r,m_r}(b_{l_1+...+l_{r-1}+1},...,b_l;c_{m_1+...+m_{r-1}+1},...,c_m)=\\\sum_{s+1}^{k+l}\sum_{k_1+...+k_s=k,l_1+...+l_s=l}\\E_{s,m}(E_{k_1l_1}(a_1,...,a_{k_1};b_1,...,b_{l_1}),...,\\E_{k_s,l_s}(a_{k_1+...+k_{s-1}+1},...,a_k;b_{l_1+...+l_{s-1}+1},...,b_l);c_1,...,c_m)\end{gathered} \tag{12}$$

We define a *Hirsch algebra* as a DG-algebra $(A,d,\cdot)$ endowed with a sequence of multioperations $\{E_{p,q}\}$ satisfying (6), (10). This name is inspired by the fact that the defining condition (10) can be regarded as generalizations of classical Hirsch formula (2). This notion was used in [12], [13].

A Hirsch algebra we call *associative* if in addition the condition (12) is satisfied.

This structure is a particular case of B_∞-algebra, see below. Moreover turn the notion of *homotopy G-algebra*, described below, is a particular case of an associative Hirsch algebra.

3.2. **Some particular cases.** For a Hirsch algebra $(A, d, \cdot, \{E_{p,q}\})$ the operation $E_{1,1} = \smile_1$ satisfies (1), so this structure can be considered as a tool which measures the noncommutativity of the product $a \cdot b$ of A. We distinguish various levels of "noncommutativity" of A according to the form of $\{E_{p,q}\}$.

Level 1. Suppose for the collection $\{E_{p,q}\}$ all the operations except $E_{0,1} = id$ and $E_{1,0} = id$ are trivial. Then it follows from (7) that in this case A is a *strictly* commutative DG-algebra.

Level 2. Suppose all operations except $E_{0,1} = id,\ E_{1,0} = id$ and $E_{1,1}$ are trivial. In this case A is endowed with a "strict" $\smile_1$ product $a \smile_1 b = E_{1,1}(a; b)$: the condition (10) here degenerate to the following 4 conditions

$$d(a \smile_1 b) = da \smile_1 b + a \smile_1 db + a \cdot b + b \cdot a,$$
$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0,$$
$$a \smile_1 (b \cdot c) + b \cdot (a \smile_1 c) + (a \smile_1 b) \cdot c = 0,$$
$$(a \smile_1 c) \cdot (b \smile_1 d) = 0.$$

The condition (12) degenerates to the associativity $\smile_1$

$$a \smile_1 (b \smile_1 c) = (a \smile_1 b) \smile_1 c.$$

As we see in this case we have very strong restrictions on $\smile_1$-product. An example of DG-algebra with such strict $\smile_1$ product is $(H^*(SX, Z_2), d = 0)$ with $a \smile_1 b = 0$ if $a \neq b$ and $a \smile_1 a = Sq^{|a|-1}a$, another example is $C^*(SX, CX)$, where SX is the suspension and CX is the cone of a space X (see [18]).

Level 3. Suppose all operations except $E_{0,1} = id,\ E_{1,0} = id$ and $E_{1,k},\ k = 1, 2, 3, ...$ are trivial. In this case the condition (10) degenerates into two conditions: at $A \otimes A^{\otimes k}$

$$\begin{gathered} dE_{1,k}(a; b_1, ..., b_k) + E_{1,k}(da; b_1, ..., b_k) + \textstyle\sum_i E_{1,k}(a; b_1, ..., db_i, ..., b_k) = \\ b_1 \cdot E_{1,k-1}(a; b_2, ..., b_k) + E_{1,k-1}(a; b_1, ..., b_{k-1}) \cdot b_k + \\ \textstyle\sum_i E_{1,k-1}(a; b_1, ..., b_i \cdot b_{i+1}, ..., b_k), \end{gathered} \tag{13}$$

and at $A^{\otimes 2} \otimes A^{\otimes k}$

$$\begin{gathered} E_{1,k}(a_1 \cdot a_2; b_1, .., b_k) = a_1 \cdot E_{1,k}(a_2; b_1, ..., b_k) + E_{1,k}(a_1; b_1, ..., b_k) \cdot a_2 + \\ \textstyle\sum_{p=1}^{k-1} E_{1,p}(a_1; b_1, ..., b_p) \cdot E_{1,m-p}(a_2; b_{p+1}, ..., b_k); \end{gathered} \tag{14}$$

moreower at $A^{\otimes n>2} \otimes A^{\otimes k}$ the condition is trivial. In particular the condition (8) here degenerates to Hirsch formula (2).

The associativity condition (12) in this case looks as

$$\begin{gathered} E_{1,n}(E_{1,m}(a; b_1, ..., b_m); c_1, ..., c_n) = \textstyle\sum_{0 \le i_1 \le ... \le i_m \le n} \sum_{0 \le n_1 + ... + n_r \le n} \\ E_{1,n-(n_1+...+n_j)+j}(a; c_1, ..., c_{i_1}, E_{1,n_1}(b_1; c_{i_1+1}, ..., c_{i_1+n_1}), c_{i_1+n_1+1}, ..., \\ c_{i_2}, E_{1,n_2}(b_2; c_{i_2+1}, ..., c_{i_2+n_2}), c_{i_2+n_2+1}, ..., \\ c_{i_m}, E_{1,n_m}(b_m; c_{i_m+1}, ..., c_{i_m+n_m}), c_{i_m+n_m+1}, ..., c_n), \end{gathered} \tag{15}$$

In particular the condition (11) here degenerates to

$$(a \smile_1 b) \smile_1 c + a \smile_1 (b \smile_1 c) = E_{1,2}(a; b, c) + E_{1,2}(a; c, b). \tag{16}$$

The structure of this level coincides with the notion of *Homotopy G-algebra*, see below.

Level 4. As the last level we consider a Hirsch algebra structure with no restrictions. An example of such structure is the cochain complex of a 1-reduced cubical set. Note that it is a *nonassociative* Hirsch algebra.

3.3. B_∞**-algebra.** The notion of $B_\infty-$algebra was introduced in [2], [10] as an additional structure on a DG-algebra $(A,\cdot,d)$ which turns the tensor coalgebra $T^c(s^{-1}A)=BA$ into a DG-bialgebra. So it requires a new differential

$$\widetilde{d}:BA\to BA$$

(which should be a coderivation with respect to standard coproduct of BA) and a new associative multiplication

$$\widetilde{\mu}:(BA,\widetilde{d})\otimes(BA,\widetilde{d})\to(BA,\widetilde{d})$$

which should be a map of DG-coalgebras, with $1_\Lambda\in\Lambda\subset BA$ as the unit element.

It is known that such $\widetilde{d}$ specifies on A a structure of A_∞-algebra in the sense of Stasheff [19], namely a sequence of operations $\{m_i:\otimes^i A\to A, i=1,2,3,...\}$ subject of appropriate conditions.

As for the new multiplication $\widetilde{\mu}$, it follows from the above considerations, that it is induced by a sequence of operations $\{E_{pq}\}$ satisfying (6), (12) and the modified condition (10) with involved A_∞-algebra structure $\{m_i\}$.

Thus the structure of associative Hirsch algebra is a particular B_∞-algebra structure on A when the standard differential of the bar construction $d_B:BA\to BA$ *does not change*, i.e. $\widetilde{d}=d_B$ (in this case the corresponding A_∞-algebra structure is degenerate: $m_1=d_A, m_2=\cdot, m_3=0, m_4=0,...$).

Let us mention, that a twisting cochain E satisfying (6) and (4), (but not (5) i.e. the induced product in the bar construction is not strictly associative), was constructed in [14] for the singular cochain complex of a topological space $C^*(X)$ using acyclic models. The condition (6) determines this twisting cochain E uniquely up to standard equivalence (homotopy) of twisting cochains in the sense of N. Berikashvili [4].

3.4. **Strong homotopy commutative algebras.** The notion of strong homotopy commutative algebra (shc-algebra), as a tool for measuring of noncommutativity of DG-algebras, was used in many papers: [17], [20], etc.

A shc-algebra is a DG-algebra $(A,d,\cdot)$ with a given twisting cochain $\Phi:B(A\otimes A)\to A$ which satisfies apropriate up to homotopy conditions of associativity and commutativity. Compare with the Hirsch algebra structure which is represented by a twisting cochain $E:BA\otimes BA\to A$. Standard contraction of $B(A\otimes A)$ to $BA\otimes BA$ allows to establish connection between these two notions.

3.5. **DG-Lie algebra structure in a Hirsch algebra.** A structure of an associative Hirsch algebra on A induces on the homology $H(A)$ a structure of Gerstenhaber algebra (G-algebra) (see [6], [8], [21]) which is defined as a commutative graded algebra $(H,\cdot)$ together with a Lie bracket of degree -1

$$[\ ,\ ]:H^p\otimes H^q\to H^{p+q-1}$$

(i.e. a graded Lie algebra structure on the desuspension $s^{-1}H$) that is a biderivation: $[a,b\cdot c]=[a,b]\cdot c+b\cdot[a,c]$.

The existence of this structure in the homology $H(A)$ is seen by the following argument.

Let $(A,d,\cdot,\{E_{p,q}\})$ be an associative Hirsch algebra, then in the desuspension $s^{-1}A$ there appears a structure of DG-Lie algebra: although the $\smile_1=E_{1,1}$ is not associative, the condition (11) implies the pre-Jacobi identity

$$a\smile_1(b\smile_1 c)+(a\smile_1 b)\smile_1 c=a\smile_1(c\smile_1 b)+(a\smile_1 c)\smile_1 b$$

This condition guarantees that the commutator $[a,b] = a \smile_1 b + b \smile_1 a$ satisfies the Jacobi identity. Besides the condition (7) implies that $[\ ,\ ] : A^p \otimes A^q \to A^{p+q-1}$ is a chain map. Thus on $s^{-1}H(A)$ there appears the structure of graded Lie algebra. The up to homotopy Hirsh formulae (8) and (9) imply that the induced Lie bracket is a biderivation.

3.6. **Homotopy G-algebra.** An associative Hirsch algebra of level 3 in the literature is known as *Homotopy G-algebra* .

A *Homotopy G-algebra* in [8] and [21] is defined as a DG-algebra $(A, d, \cdot)$ with a given sequence of multibraces $a\{a_1, ..., a_k\}$ which, in our notation, we regard as a sequence of operations

$$E_{1,k} : A \otimes (\otimes^k A) \to A, \quad k = 0, 1, 2, 3, ...$$

which, together with $E_{01} = id$ satisfies the conditions (6), (13), (14) and (15).

The name *Homotopy G-algebra* is motivated by the fact that this structure induces on the homology $H(A)$ the structure of G-algebra (as we have seen in the previous section such a structure appears even on the homology of an associative Hirsch algebra).

The conditions (13), (14), and (15) in [8] are called *higher homotopies*, *distributivity* and *higher pre-Jacobi identities* respectively. As we have seen the first two conditions mean that $E : BA \otimes BA \to A$ is a twisting cochain, or equivalently $\mu_E : BA \otimes BA \to BA$ is a chain map, and the third one means that this multiplication is associative.

3.7. **Operadic description.** Appropriate language to describe such huge sets of operations is the operadic language. Here we use *surjection operad* χ and *Barratt-Eccles operad* $\mathcal{E}$ which are most convenient E_∞ operads. For definitions we refer to [3].

The operations $E_{1,k}$ forming hGa have nice description in the *surjection operad*, see [15], [16], [3]. Namely, to the dot product corresponds the element $(1,2) \in \chi_0(2)$, to $E_{1,1} = \smile_1$ product corresponds $(1,2,1) \in \chi_1(2)$, to the operation $E_{1,2}$ the element $(1,2,1,3) \in \chi_2(3)$, etc. Generally to the operation $E_{1,k}$ corresponds the element

$$E_{1,k} = (1, 2, 1, 3, ..., 1, k, 1, k+1, 1) \in \chi_k(k+1). \tag{17}$$

We remark here that the defining conditions of a hGa (13), (14), (15) can be expressed in terms of operadic structure (differential, symmetric group action and composition product) and the elements (17) satisfy these conditions *already in the operad* χ. This in particular implies that *any* χ*-algebra is automatically a hGa.* Note that the elements (17) together with $(1,2)$ generate the suboperad $F_2\chi$ which is equivalent to the little square operad. This fact and a hGa structure on the Hochschild cochain complex $C^*(U,U)$ of an algebra U are used by many authors to prove so called Deligne conjecture about the action of the little square operad on $C^*(U,U)$.

Now look at the operations $E_{p,q}$ which define a structure of Hirsch algebra. They *can not live* in χ: it is enough to mention that the Hirsch formula (2), as a part of defining conditions of hGa, is satisfied in χ, but for a Hirsch algebra this condition is satisfied up to homotopy $E_{2,1}$, see (8). We belive that $E_{p,q}$-s live in

the Barratt-Eccles operad $\mathcal{E}$. In particular direct calculation shows that

$$\begin{aligned}
&E_{1,1} = ((1,2),(2,1)) \in \mathcal{E}_1(2);\\
&E_{1,2} = ((\mathbf{1},2,3),(2,\mathbf{1},3),(2,3,\mathbf{1})) \in \mathcal{E}_2(3);\\
&E_{2,1} = ((1,2,\mathbf{3}),(1,\mathbf{3},2),(\mathbf{3},1,2)) \in \mathcal{E}_2(3);\\
&E_{1,3} = ((\mathbf{1},2,3,4),(2,\mathbf{1},3,4),(2,3,\mathbf{1},4),(2,3,4,\mathbf{1})) \in \mathcal{E}_3(4);\\
&E_{3,1} = ((1,2,3,\mathbf{4}),(1,2,\mathbf{4},3),(1,\mathbf{4},2,3),(\mathbf{4},1,2,3)) \in \mathcal{E}_3(4);
\end{aligned}$$

and in general

$$\begin{aligned}
E_{1,k} &= ((\mathbf{1},2,...,k+1),...,(2,3,...,i,\mathbf{1},i+1,...,k+1),...,(2,3,...,k+1,\mathbf{1}));\\
E_{k,1} &= ((1,2,...,\mathbf{k+1}),...,(1,2,...,i,\mathbf{k+1},i+1,...,k),...,(\mathbf{k+1},1,2,...,k)).
\end{aligned}$$

As for other $E_{p,q}$-s we can indicate just

$$\begin{aligned}
E_{2,2} = \quad & ((1,2,3,4),(1,3,4,2),(3,1,4,2),(3,4,1,2))+\\
& ((1,2,3,4),(3,1,2,4),(3,1,4,2),(3,4,1,2))+\\
& ((1,2,3,4),(1,3,2,4),(1,3,4,2),(3,1,4,2))+\\
& ((1,2,3,4),(1,3,2,4),(3,1,2,4),(3,1,4,2)).
\end{aligned}$$

We remark that the operadic *table reduction* map $TR : \mathcal{E} \to \chi$, see [3], maps $E_{k>1,1}$ and $E_{2,2}$ to zero, and $E_{1,k} \in \mathcal{E}_k(k+1)$ to $E_{1,k} \in \chi_k(k+1)$.

4. Adams $\smile_1$-product in the cobar construction of a bialgebra

Here we present the Adams $\smile_1$-product $\smile_1$: $\Omega A \otimes \Omega A \to \Omega A$ on the cobar construction ΩA of a DG-bialgebra $(A, d, \Delta : A \to A \otimes A, \mu : A \otimes A \to A)$ (see [1]). This will be the first step in the construction of hGa structure on ΩA.

This $\smile_1$ product satisfies the Steenrod condition (1) and the Hirsch formula (2).

First we define the $\smile_1$-product of two elements $x = [a], y = [b] \in \Omega A$ of length 1 as $[a] \smile_1 [b] = [a \cdot b]$. Extending this definition by (2) we obtain

$$\begin{gathered}
[a_1, a_2] \smile_1 [b] = ([a_1] \cdot [a_2]) \smile_1 [b] = [a_1] \cdot ([a_2] \smile_1 [b]) + ([a_1] \smile_1 [b]) \cdot [a_2] =\\
[a_1] \cdot [a_2 \cdot b] + [a_1 \cdot b] \cdot [a_2] = [a_1, a_2 \cdot b] + [a_1 \cdot b, a_2].
\end{gathered}$$

Further iteration of this process gives

$$[a_1, ..., a_n] \smile_1 [b] = \sum_i [a_1, ..., a_{i-1}, a_i \cdot b, a_{i+1}, ..., a_n].$$

Now let's define $[a] \smile_1 [b_1, b_2] = [a^{(1)} \cdot b, a^{(2)} \cdot b]$ where $\Delta a = a^{(1)} \otimes a^{(2)}$ is the value of the diagonal $\Delta : A \to A \otimes A$ on $[a]$. Inspection shows that the condition (1) for short elements

$$d([a] \smile_1 [b]) = d[a] \smile_1 [b] + [a] \smile_1 d[b] + [a] \cdot [b] + [b] \cdot [a].$$

is satisfied.

Generally we define the $\smile_1$ product of an element $x = [a] \in \Omega A$ of length 1 and an element $y = [b_1, ..., b_n] \in \Omega A$ of arbitrary length by

$$[a] \smile_1 [b_1, ..., b_n] = [a^{(1)} \cdot b_1, ..., a^{(n)} \cdot b_n],$$

here $\Delta^n(a) = a^{(1)} \otimes ... \otimes a^{(n)}$ is the n-fold iteration of the diagonal $\Delta : A \to A \otimes A$ and $a \cdot b = \mu(a \otimes b)$ is the product in A.

Extending this definition for the elements of arbitrary lengths $[a_1,...,a_m] \smile_1 [b_1,...,b_n]$ by the Hirsch formula (2) we obtain the general formula

$$[a_1,...,a_m] \smile_1 [b_1,...,b_n] = \sum_k [a_1,...,a_{k-1}, a_k^{(1)} \cdot b_1, ..., a_k^{(n)} \cdot b_n, a_{k+1}, ..., a_m]. \tag{18}$$

Of course so defined $\smile_1$ satisfies the Hirsch formula (2) automatically. It remains to prove the

Proposition 2. *This $\smile_1$ satisfies Steenrod condition (1)*

$$\begin{gathered} d_\Omega([a_1,...,a_m] \smile_1 [b_1,...,b_n]) = \\ d_\Omega[a_1,...,a_m] \smile_1 [b_1,...,b_n] + [a_1,...,a_m] \smile_1 d_\Omega[b_1,...,b_n] + \\ [a_1,...,a_m,b_1,...,b_n] + [b_1,...,b_n,a_1,...,a_m]. \end{gathered}$$

Proof. Let us denote this condition by $Steen_{m,n}$. The first step consists in direct checking of the conditions $Steen_{1,m}$ by induction on m. Furthermore, assume that $Steen_{m,n}$ is satisfied. Let us check the condition $Steen_{m+1,n}$ for $[a,a_1,...,a_m] \smile_1 [b_1,...,b_n]$. We denote $[a_1,...,a_m] = x,\ [b_1,...,b_m] = y$. Using the Hirsch formula (2), $Steen_{m,n}$, and $Steen_{1,n}$ we obtain:

$$\begin{gathered} d([a,a_1,...,a_m] \smile_1 [b_1,...,b_n]) = \\ d(([a]\cdot x) \smile_1 y) = d([a]\cdot (x \smile_1 y) + ([a] \smile_1 y)\cdot x) = \\ = d[a]\cdot (x \smile_1 y) + [a]\cdot (dx \smile_1 y + x \smile_1 dy + x\cdot y + y\cdot x) + \\ (d[a] \smile_1 y + [a] \smile_1 dy + [a]\cdot y + y \cdot [a])\cdot x + ([a] \smile_1 y)dx = \\ d[a]\cdot (x \smile_1 y) + [a]\cdot (dx \smile_1 y) + [a]\cdot (x \smile_1 dy) + [a]\cdot x\cdot y + [a]\cdot y \cdot x + \\ (d[a] \smile_1 y\cdot)x + ([a] \smile_1 dy\cdot)x + [a]\cdot y\cdot x + y\cdot [a]\cdot x + ([a] \smile_1 y)dx. \end{gathered}$$

Besides, using Hirsch (2) formula we obtain

$$\begin{gathered} d[a,a_1,...,a_m] \smile_1 [b_1,...,b_n] = \\ d([a]\cdot x) \smile_1 y = (d[a]\cdot x) \smile_1 y + ([a]\cdot dx) \smile_1 y = \\ d[a]\cdot (x \smile_1 y) + (d[a] \smile_1 y)\cdot x + [a]\cdot (dx \smile_1 y) + ([a] \smile_1 y)\cdot dx \end{gathered}$$

and

$$\begin{gathered} [a,a_1,...,a_m] \smile_1 d[b_1,...,b_n] = \\ ([a]\cdot x) \smile_1 dy = [a]\cdot (x \smile_1 dy) + ([a] \smile_1 dy)\cdot x, \end{gathered}$$

now it is evident that $Steen_{m+1,n}$ is satisfied. This completes the proof.

5. Homotopy G-algebra structure on the cobar construction of a bialgebra

Below we present a sequence of operations

$$E_{1,k} : \Omega A \otimes (\Omega A)^{\otimes k} \to \Omega A,$$

which extends the above described $E_{1,1} = \smile_1$ to a structure of homotopy G-algebra on the cobar construction of a DG-bialgebra . This means that $E_{1,k}$-s satisfy the conditions (13), (14) and (15).

For $x = [a] \in \Omega A$ of length 1, $y_i \in \Omega A$ and $k > 1$ we define $E_{1,k}([a]; y_1, ...y_k) = 0$ and extend for an arbitrary $x = [a_1, ...a_n]$ by (14). This gives

$$E_{1,k}([a_1,...,a_n]; y_1,...,y_k) = 0$$

for $n < k$ and

$$E_{1,k}([a_1,...,a_k]; y_1,...,y_k) = [a_1 \diamond y_1, ..., a_k \diamond y_k],$$

here we use the notation $a\diamond(b_1,...,b_s)=(a^{(1)}\cdot b_1,...,a^{(s)}\cdot b_s)$, so using this notation $[a]\smile_1[b_1,...,b_s]=[a\diamond(b_1,...,b_s)]$. Further iteration by (14) gives the general formula

$$\begin{gathered}E_{1,k}([a_1,...,a_n];y_1,...,y_k)=\\ \sum[a_1,...,a_{i_1-1},a_{i_1}\diamond y_1,a_{i_1+1},...,a_{i_k-1},a_{i_k}\diamond y_k,a_{i_k+1},...,a_n],\end{gathered}\tag{19}$$

where the summation is taken over all $1\leq i_1<...<i_k\leq n$.

Of course so defined operations $E_{1,k}$ automatically satisfy the condition (14). It remains to prove the

Proposition 3. *The operations $E_{1,k}$ satisfy the conditions (13) and (15).*

Proof. The condition (13) is trivial for $x=[a]$ of length 1 and $k>2$. For $x=[a]$ and $k=2$ this condition degenerates to

$$E_{1,1}([a];y_1\cdot y_2)+y_1\cdot E_{1,1}([a];y_2)-E_{1,1}([a];y_1)\cdot y_2=0$$

and this equality easily follows from the definition of $E_{1,1}=\smile_1$. For a long $x=[a_1,...,a_m]$ the condition (13) can be checked by induction on the length m of x using the condition (14).

Similarly, the condition (15) is trivial for $x=[a]$ of length 1 unless the case $m=n=1$ and in this case this condition degenerates to

$$E_{1,1}(E_{1,1}(x;y);z)=E_{1,1}(x;E_{1,1}(y);z))+E_{1,2}(x;y,z)+E_{1,2}(x;z,y).$$

This equality easily follows from the definition of $E_{1,1}=\smile_1$. For a long $x=[a_1,...,a_m]$ the condition (15) can be checked by induction on the length m of x using the condition (14).

Remark 1. *For a DG-coalgebra $(A,d,\Delta:A\to A\otimes A)$ there is a standard DG-coalgebra map $g_A:A\to B\Omega A$ from A to the bar of cobar of A. This map is the coextension of the universal twisting cochain $\phi_A:A\to\Omega A$ defined by $\phi(a)=[a]$ and is a weak equivalence, i.e. it induces an isomorphism of homology. Suppose A is a DG-bialgebra. Then the constructed sequence of operations $E_{1,k}$ define a multiplication $\mu_E:B\Omega A\otimes B\Omega A\to B\Omega A$ on the bar construction $B\Omega A$ so that it becomes a DG-bialgebra. Direct inspection shows that $g_A:A\to B\Omega A$ is multiplicative, so it is a weak equivalence of DG-bialgebras. Dualizing this statement we obtain a weak equivalence of DG-bialgebras $\Omega BA\to A$ which can be considered as a free (as an algebra) resolution of a DG-bialgebra A.*

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